Tutorial 4 (10 Feb)

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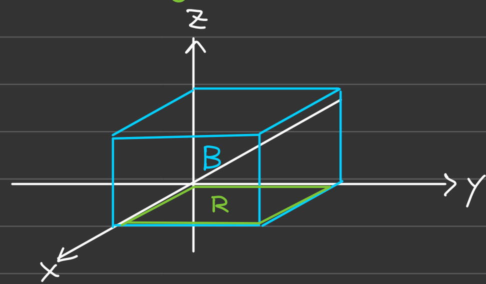
Fubini's Theorem for triple integrals

Thm 1 (Fubini's Theorem for continuous functions over rectangular boxes)

Let $g : B \to \mathbb{R}$ be a continuous function over a solid B, where

- $B := [a,b] \times [c,d] \times [e,f] = R \times [e,f] \subseteq \mathbb{R}^3$ is a rectangular box, where
- $R := [a,b] \times [c,d] \subseteq \mathbb{R}^2$ is a rectangle.



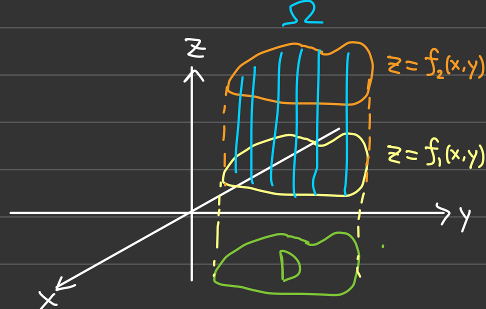
then
$$\iiint_B g \, dV = \iint_R \left(\int_e^f g(x,y,z)\, dz \right) dA(x,y)$$
$$= \int_a^b \int_c^d \int_e^f g(x,y,z)\, dz\, dy\, dx$$

Thm 2 (Fubini's Theorem for continuous functions over more general solids)

Let $g : \Omega \to \mathbb{R}$ be a continuous function over a solid Ω, where

- $\Omega := \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D \,;\, f_1(x,y) \le z \le f_2(x,y)\} \subseteq \mathbb{R}^3$, where
  - $D \subseteq \mathbb{R}^2$ is a region.
  - $f_1, f_2 : D \to \mathbb{R}$ are continuous

    with $f_1(x,y) \le f_2(x,y)$, $\forall (x,y) \in D$.



then
$$\iiint_\Omega g \, dV = \iint_D \left(\int_{f_1(x,y)}^{f_2(x,y)} g(x,y,z)\, dz \right) dA(x,y)$$

Rmk Similar formulae for other orders of variables, e.g. $dx\,dy\,dz$, ….

Fubini's Theorem for triple integrals in cylindrical coordinates

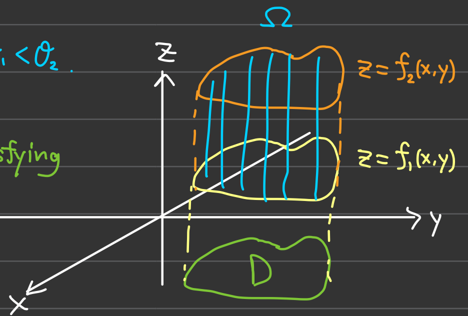
Cor (Fubini's Theorem for continuous functions in cylindrical coordinates)

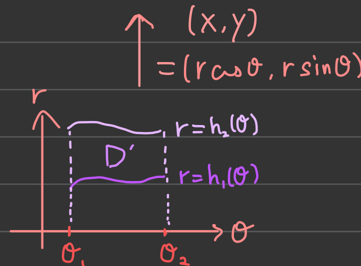
Let $g : \Omega \to \mathbb{R}$ be a continuous function over a solid Ω, where

- $\Omega := \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D \,;\, f_1(x,y) \le z \le f_2(x,y)\} \subseteq \mathbb{R}^3$, where
  - $D = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid \theta_1 \le \theta \le \theta_2 \,;\, h_1(\theta) \le r \le h_2(\theta)\}$, where
    - $\theta_1, \theta_2 \in [0, 2\pi)$ are constants satisfying $\theta_1 < \theta_2$.
    - $h_1, h_2 : [\theta_1, \theta_2] \to \mathbb{R}$ are continuous satisfying

      $0 \le h_1(\theta) \le h_2(\theta)$ for any $\theta \in [\theta_1, \theta_2]$.
  - $f_1, f_2 : D \to \mathbb{R}$ are continuous

    with $f_1(x,y) \le f_2(x,y)$, $\forall (x,y) \in D$





then
$$\iiint_\Omega g \, dV = \iint_D \left(\int_{f_1(x,y)}^{f_2(x,y)} g(x,y,z)\, dz \right) dA(x,y)$$
$$= \int_{\theta_1}^{\theta_2} \int_{h_1(\theta)}^{h_2(\theta)} \int_{f_1(r\cos\theta, r\sin\theta)}^{f_2(r\cos\theta, r\sin\theta)} g(r\cos\theta, r\sin\theta, z)\, r \, dz\, dr\, d\theta$$

Cor (Volume of a solid via a triple integral in cylindrical coordinates)

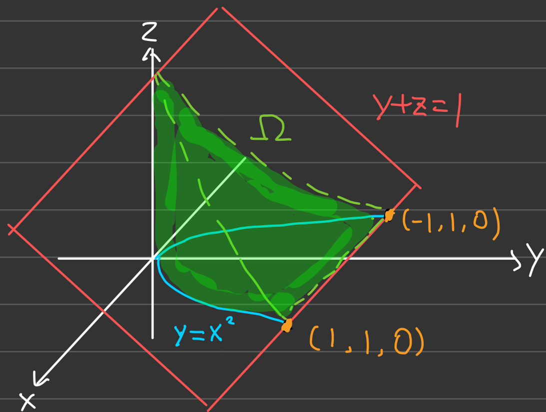
Given a solid Ω as above, then its volume is

$$\mathrm{Vol}(\Omega) \overset{\text{Def}}{:=} \iiint_\Omega 1 \cdot dV \overset{\text{Thm}}{=} \int_{\theta_1}^{\theta_2} \int_{h_1(\theta)}^{h_2(\theta)} \int_{f_1(r\cos\theta, r\sin\theta)}^{f_2(r\cos\theta, r\sin\theta)} r \, dz\, dr\, d\theta.$$

Ex Find the volume of the solid enclosed by the cylinder $y=x^2$ and the planes $z=0$, $y+z=1$.

Sol Idea: Compute the volume by a triple integral

Step 1 Sketch the solid Ω.



Step 2 Describe Ω in Cartesian coordinates.

$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; -1 \le x \le 1,\ x^2 \le y \le 1,\ 0 \le z \le 1-y \right\}$$

Step 3 Compute the volume of Ω by a triple integral.

$$\mathrm{Vol}(\Omega) = \iiint_D 1 \cdot dV = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} [z]_0^{1-y}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (1-y)\,dy\,dx = \int_{-1}^{1} \left[y - \frac{y^2}{2} \right]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \left(\left(1-\frac{1}{2}\right) - \left(x^2 - \frac{x^4}{2}\right) \right) dx = \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx$$

$$= 2\int_0^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx = 2\left[\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10} \right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10} \right) = \frac{8}{15}$$

Rmk Alternatively, the volume can be computed by a double integral

$$\iint_D (1-y)\,dA, \text{ where } D = \{ (x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1,\ x^2 \le y \le 1 \}.$$